An Undergraduate's take on: Quantum Entanglement vs Local Realism and closing of the Detection Loophole

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Abstract

Non locality and uncertainty of the quantum theory was a matter of huge debate for a very long time. Einstein, Podolsky and Rosen introduced local realism which was more like a pre-deterministic idea. It said that the particles that are found in certain quantum states are the one which were in that state from the beginning. Later on J.Bell did an experiment and found out that the hypothesis of EPR was rather wrong. Though the idea of local realism was wrong it motivated many people to look from different perspectives. There are two major questions raised by the local realism on the quantum mechanics one is known as the locality loophole and the other one is detection loophole. In this paper I have discussed about the Bell's inequalities and how it violates the idea of local realism, CHSH inequalities and also the efficiency required by the detectors to close the detection loophole.

1 Introduction

Einstein from the very beginning was not very comfortable with the idea proposed by the Quantum mechanics where the future was uncertain. He was quite famous for saying

> *"God might be sophisticated but he does not play dice."*

Not that he was uncomfortable with the introduction of probabilities in physics but to him the idea that the God does not know about the future or future was uncertain, was rather ridiculous.

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In 1935, Einstein along with Podolsky and Rosen[1] argued that Quantum Mechanics cannot be a complete theory of the physical world. They based their argument on a phenomena called Quantum Entanglement. According to Entanglement theory if two particles are entangled the characteristic or the state of one particle can be determined by measuring other particle, assuming the measurement did not affected the other particle. In other words suppose we have an entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+\rangle\,|-\rangle + |-\rangle\,|+\rangle)$$

QM says that if we measure the first particle the wave function collapses into one of the states depending on what the state of the first particle was. EPR then argued that there is no such collapse of the wave function as we measure. The states in which the particles are found to be are the states they were in from the beginning. We cannot find the state in which they are before measuring is our mere ignorance. We do not have the complete picture to lay out the mathematical model of the particles. This idea of EPR was then known to be local realism. EPR also claimed that there must be so called hidden variables which we do not know but it exists, which has all the information about the particle.

In 1964 John Bell published a related thought experiment[2] which distinguished between QM and local realism as they predicted statistically different experimental results.

In this paper I have given a layout of Bell's experiment using photons, discussed CHSH inequality and using that closed the detection loophole created by local realism.

2 EPR paradox and Bell inequalities

The argument of Einstein, Podolsky and Rosen that the universe is dictated by the so called hidden variables was unchallenged for about 30 years. In year 1964 John Bell devised a thought experiment to test this theory. Although Bell in his paper [2] used spin 1/2 particles but in experiments photons are easier to work with, so here I will use the photons to recreate the Bell's inequality.

Usually entangled photons are created using Spontaneous Parametric Down Conversion (SPDC)[3]. Here we will use Type-II SPDC photons in which the entangled photons have different polarization if measured along same axis.

2.1 Using Hidden variable Theory

Let λ be the hidden variable that decides the polarization of the photons received by Alice(A) and Bob(B).

Let's say,
Alice's photon $\rightarrow$ A($\vec{a}$,λ)
Bob's photon $\rightarrow$ B($\vec{b}$,λ), are maximally entangled.
Now, suppose horizontal polarization takes value : +1
and , vertical polarization takes value : -1
then we can say

$$A(\vec{a},\lambda) = \pm 1 \ ; \ B(\vec{b},\lambda) = \pm 1 \tag{1}$$

Since, photons are entangled using Type-II SPDC so they must have opposite polarization if measured along the same axis. i.e.

$$A(\vec{a},\lambda) = -B(\vec{a},\lambda) \tag{2}$$

Calculating expectation value of Alice's and Bob's photon

$$\langle\psi| A(\vec{a},\lambda)B(\vec{b},\lambda) |\psi\rangle = \int \rho(\lambda)A(\vec{a},\lambda)B(\vec{b},\lambda)d\lambda$$

where $\rho(\lambda)$ is probability density of the hidden variable
using equation (1) we can write
B($\vec{b}$,λ)= -A($\vec{b}$,λ)
Introducing another polarizer direction $\vec{c}$ in which bob measures and this time calculating the difference of expectation values.

$$\begin{aligned} &\langle\psi| A(\vec{a},\lambda)B(\vec{b},\lambda) |\psi\rangle - \langle\psi| A(\vec{a},\lambda)B(\vec{c},\lambda) |\psi\rangle \\ &= -\int \rho(\lambda)A(\vec{a},\lambda)A(\vec{b},\lambda)d\lambda - \int \rho(\lambda)A(\vec{a},\lambda)B(\vec{c},\lambda)d\lambda \\ &= -\int \rho(\lambda)(A(\vec{a},\lambda)A(\vec{b},\lambda) - A(\vec{a},\lambda)A(\vec{c},\lambda))d\lambda \end{aligned}$$

since we know (A($\vec{b}$,λ))^2 =1, we can write

$$= -\int \rho(\lambda)A(\vec{a},\lambda)A(\vec{b},\lambda)[A(\vec{b},\lambda)^2 - A(\vec{b},\lambda)A(\vec{c},\lambda)]d\lambda$$

Taking absolute values

$$\begin{aligned} &|\langle\psi| A(\vec{a},\lambda)B(\vec{b},\lambda) |\psi\rangle - \langle\psi| A(\vec{a},\lambda)B(\vec{c},\lambda) |\psi\rangle| \\ &= \int \rho(\lambda)A(\vec{a},\lambda)A(\vec{b},\lambda)(1 - A(\vec{b},\lambda)A(\vec{c},\lambda))d\lambda \end{aligned}$$

We can say that A($\vec{a}$,λ) A($\vec{b}$,λ) = $\pm$ 1
if we take the value to be +1 then we would be overestimating the RHS thus creating an inequality

$$\begin{aligned} &\leq \int \rho(\lambda)(1 - A(\vec{b}, \lambda)A(\vec{c}, \lambda))d\lambda \\ &= 1 - \langle\psi| A(\vec{b}, \lambda)A(\vec{c}, \lambda) |\psi\rangle \end{aligned}$$

So, the Bell's inequality is

$$\begin{aligned} &| \langle\psi| A(\vec{a}, \lambda)B(\vec{b}, \lambda) |\psi\rangle - \langle\psi| A(\vec{a}, \lambda)B(\vec{c}, \lambda) |\psi\rangle | \\ &\leq 1 - \langle\psi| A(\vec{b}, \lambda)A(\vec{c}, \lambda) |\psi\rangle \end{aligned}$$

which can also be written as

$$|\vec{a}.\vec{b} - \vec{a}.\vec{c}| \leq 1 - \vec{b}.\vec{c}$$

which is proved in next section.

3 Clauser-Horne-Shimony-Holt (CHSH) inequality

CHSH inequality[4] was derived in 1969 in accordance with original Bell's inequality which is true if there exist underlying local hidden variables.
To derive this let's suppose Alice measures her photon in two different directions $\vec{a}$ and $\vec{a'}$ and Bob measures in directions $\vec{b}$ and $\vec{b'}$.
From equation (1)

$$A(\vec{a}, \lambda) = A'(\vec{a'}, \lambda) = B(\vec{b}, \lambda) = B'(\vec{b'}, \lambda) = \pm 1$$

For Alice if A+A′=0 then we can say that A-A′=$\pm$2
Also
For Bob if B+B′=0 then we can say that B-B′=$\pm$2
Let's define

$$C = B(A + A') + B'(A - A') \tag{3}$$

This step is driven by the hidden variable theory because with Quantum Mechanics neither Alice nor Bob can measure simultaneously along two different directions. But here we've assumed ± 1 for all of them.
A common inequality in Mathematics would suggest that

$$| \langle\psi| C |\psi\rangle | \leq \langle\psi| |C| |\psi\rangle = 2$$

By expanding it we can write

$$| \langle\psi| AB |\psi\rangle + \langle\psi| A'B |\psi\rangle + \langle\psi| AB' |\psi\rangle - \langle\psi| A'B' |\psi\rangle | \leq 2 \tag{4}$$

This is known as the CHSH inequality.
Now, checking whether Quantum Mechanics follows the inequality

$$\begin{aligned} Let \quad & A = \boldsymbol{\sigma^{(A)}}.\hat{a} \qquad A' = \boldsymbol{\sigma^{(A)}}.\hat{a'} \\ & B = \boldsymbol{\sigma^{(B)}}.\hat{b} \qquad B' = \boldsymbol{\sigma^{(B)}}.\hat{b'} \end{aligned}$$

which are all hermitian operators So,

$$\begin{aligned} \langle\psi| AB |\psi\rangle &= \langle\psi| (\boldsymbol{\sigma^{(A)}}.\hat{\boldsymbol{a}})(\boldsymbol{\sigma^{(B)}}.\hat{b}) |\psi\rangle \\ &= -\langle\psi| (\boldsymbol{\sigma^{(A)}}.\hat{a})(\boldsymbol{\sigma^{(A)}}.\hat{b}) |\psi\rangle \\ &= -a_i b_j \langle\psi| \sigma_i^{(A)} \sigma_j^{(A)} |\psi\rangle \\ &= -a_i b_j \delta_{ij} \\ &= -\hat{a}.\hat{b} = -cos\theta \end{aligned}$$

where θ is the angle between $\hat{a}$ and $\hat{b}$

From Equation (2):
A($\vec{a}$,λ) = - B($\vec{b}$,λ)
$\boldsymbol{\sigma^{(A)}}.\hat{a}$ = - $\boldsymbol{\sigma^{(B)}}.\hat{b}$
$\boldsymbol{\sigma^{(A)}}$ = - $\boldsymbol{\sigma^{(B)}}$
Similarly,

$$\langle\psi| A'B |\psi\rangle = -\hat{a'}.\hat{b} \qquad \langle\psi| AB' |\psi\rangle = -\hat{a}.\hat{b'} \qquad \langle\psi| A'B' |\psi\rangle = -\hat{a'}.\hat{b'} \tag{5}$$

choosing $\hat{a'}$,$\hat{a}$,$\hat{b}$,$\hat{b'}$ such that they are separated by 45° consecutively
From the inequality (4) we get

$$|\frac{-4}{\sqrt{2}}| \leq 2 \qquad or \quad 2\sqrt{2} \leq 2$$

Thus Quantum Mechanics(QM) violates the CHSH inequalities. And since by choosing angle as 45° QM violates the inequality by maximum amount, therefore this is called maximal violation. The inequality which is in accordance to the quantum theory is known as Cirel'son's inequality [5].

Coming back to CHSH inequality from (4) and (5) we can write

$$|-\hat{a}.\hat{b} - \hat{a'}.\hat{b} - \hat{a}.\hat{b'} + \hat{a'}.\hat{b'}| \leq 2 \tag{6}$$

Since,

$$-1 \leq \hat{a}.\hat{b} = cos\theta \leq 1$$

Therefore the inequality does not get affected if we choose $\hat{a}$, $\hat{a'}$, $\hat{b}$, $\hat{b'}$={0,1}
If we take $\hat{a}$,$\hat{a'}$,$\hat{b}$,$\hat{b'}$ such that they maximally violate CHSH inequality then,
-$\hat{a}.\hat{b}$ - $\hat{a'}.\hat{b}$ - $\hat{a}.\hat{b'}$+ $\hat{a'}.\hat{b'}$ is negative . Using it in inequality (6)

$$-(-\hat{a}.\hat{b} - \hat{a'}.\hat{b} - \hat{a}.\hat{b'} + \hat{a'}.\hat{b'}) \leq 2$$

Slightly modifying the inequality we can write

$$\hat{a}.\hat{b} + \hat{a'}.\hat{b} + \hat{a}.\hat{b'} - \hat{a'}.\hat{b'} \leq \hat{a} + \hat{b}$$

One can check the validity of this inequality by running a case bash having 16 possible cases.
Expanding it and dividing it by N on both sides

$$\frac{1}{N}\sum_i a_i b_i + a'_i b_i + a_i b'_i - a'_i b'_i \leq \frac{1}{N}\sum_i a_i + b_i$$

Which is the expression of probability. So, using CHSH inequality we got

$$P_{++}(a,b) + P_{++}(a',b) + P_{++}(a,b') - P_{++}(a',b') \leq P_{+.}(a) + P_{.+}(b) \quad (7)$$

where $P_{++}(a,b)$ is the probability that Alice's detector clicks if aligned along $\hat{a}$ and Bob's detector clicks if aligned along $\hat{b}$ and $P_{+.}(a)$ is probability that Alice's detector clicks if aligned along $\hat{a}$. Given detectors have 100% efficiency.

4 Loopholes in Quantum Mechanics suggested by local realism

There are mainly three types of loopholes that are found by the local realism Locality loophole, Detection loophole and free will loophole. In free will loophole it is said that Alice's and Bob's decision is governed by hidden variables. But usually this is not taken as seriously as other two.

4.1 Locality loophole

When we were deriving the CHSH inequality, we assumed that Alice can measure in either $\hat{a}$ or $\hat{a'}$ direction and if Alice goes first then no information about Alice's measurement reaches Bob before he measures his own and vice versa. But if the information about Alice's measurement reaches Bob before he measures then the marginal probability distribution of Bob gets updated and CHSH inequality might not be true.This is known as Locality loophole.

The assumption we took about Information not reaching Bob is backed up by relativistic causality if Bob and Alice are spacelike separated. In 1982, Alain Aspect conducted an experiment [6] to address the locality loophole. In this two entangled photons and detectors were used which were spacelike separated. The results were consistent with Quantum predictions and violated CHSH inequality.

4.2 Detection loophole

While experimenting with photons what happens is sometimes if detection efficiency is too low then many entangled photons are not detected either in Alice's detector or Bob's detector or neither one of them. But the data we accept from the experiment only when photons are detected in both the detectors. This might result in the reduction of sample space and the data collected might be biased.

What local realism says to this is what if the hidden variables not only control the polarization state of the photon but also determines if the photon in the detector is going to get detected at all or not.

So, what should be the efficiency of the detectors such that the detection loophole could be easily closed for a maximally entangled photon state.

5 Closing of the detection loophole

In equation (7) we've taken the detectors to be perfect but in reality this does not happen. So, we write

$$P_{Alice} = \eta_{(A)} P_{perf} \qquad P_{Bob} = \eta_{(B)} P_{perf}$$

Where, P_{Alice} is probability of Alice's imperfect detector to detect a photon and $\eta_{(A)}$ is efficiency of the detector.

Rewriting equation (7) with imperfect detectors and expanding in terms of sum.

$$\eta_{(A)}\eta_{(B)} N^{-1} \sum_i a_i b_i + a'_i b_i + a_i b'_i - a'_i b'_i \leq N^{-1} \sum_i \eta_{(A)} a_i + \eta_{(B)} b_i \tag{8}$$

Since, here we are using the photon states and polarization of photon states are analogous to spins of spin-1/2 particles like electrons [7]

So, the projection operator that we use for spin 1/2 particles can also be used here.

The projection operator in matrix form

$$E(\hat{n}, \pm) = \frac{1}{2}(I \pm \hat{n}.\boldsymbol{\sigma})$$

Where + is for horizontal polarization and - is for vertical polarization

Writing probabilities as expectation value of product of projection operators. since, entangled photons are oppositely polarized, i.e. if photon A is horizontally polarized then photon B will be vertically polarized and vice versa.

$$\begin{aligned} P_{++}(ab) &= \eta_{(A)}\eta_{(B)} \langle\psi| E(\hat{a}, +) E(\hat{b}, -) |\psi\rangle \\ &= \frac{\eta_{(A)}\eta_{(B)}}{4} \langle\psi| (I + \hat{a}.\boldsymbol{\sigma^{(A)}})(I - \hat{b}.\boldsymbol{\sigma^{(B)}}) |\psi\rangle \end{aligned} \tag{9}$$

From equation (3)

$$\boldsymbol{\sigma}^{(\boldsymbol{A})} = -\boldsymbol{\sigma}^{(\boldsymbol{B})}$$

So, equation (9) becomes

$$\begin{aligned} P_{++}(ab) &= \frac{\eta_{(A)}\eta_{(B)}}{4}(1 + \langle\psi| \, (\hat{a}.\boldsymbol{\sigma}^{(\boldsymbol{A})})(\hat{b}.\boldsymbol{\sigma}^{(\boldsymbol{A})}) \, |\psi\rangle \\ &= \frac{\eta_{(A)}\eta_{(B)}}{4}(1 + \hat{a}.\hat{b}) \end{aligned}$$

Similarly,

$$\begin{aligned} P_{++}(a'b) &= \frac{\eta_{(A)}\eta_{(B)}}{4}(1 + \hat{a'}.\hat{b}) \\ P_{++}(ab') &= \frac{\eta_{(A)}\eta_{(B)}}{4}(1 + \hat{a}.\hat{b'}) \\ P_{++}(a'b') &= \frac{\eta_{(A)}\eta_{(B)}}{4}(1 + \hat{a'}.\hat{b'}) \end{aligned}$$

In RHS of the equation (8)

Since, Alice measures half of the time along $\hat{a}$ and other half of the time along $\hat{a'}$. So, the probability that Alice's detector clicks if oriented along $\hat{a}$ is $\frac{efficiency\ of\ detector}{2}$.

i.e.

$$P_{+.}(a) = \frac{\eta_{(A)}}{2}$$

Similarly,

$$P_{.+}(b) = \frac{\eta_{(B)}}{2}$$

Putting everything in inequality (8)

$$\eta_{(A)}\eta_{(B)}(\frac{1}{2} + \frac{1}{4}(\hat{a}.\hat{b} + \hat{a'}.\hat{b} + \hat{a}.\hat{b'} - \hat{a'}.\hat{b'}) \leq \frac{\eta_{(A)} + \eta_{(B)}}{2}$$

For maximal violation take $\hat{a'}$,$\hat{b}$,$\hat{a}$,$\hat{b'}$ to be co-planar and separated by 45° then,

$$\begin{aligned} \eta_{(A)}\eta_{(B)}(\frac{1}{2} + \frac{1}{\sqrt{2}}) &\leq \frac{\eta_{(A)} + \eta_{(B)}}{2} \\ \Rightarrow \frac{\eta_{(A)}\eta_{(B)}}{2}(1 + \sqrt{2}) &\leq \frac{\eta_{(A)} + \eta_{(B)}}{2} \\ \Rightarrow \frac{\eta_{(A)}\eta_{(B)}}{\eta_{(A)} + \eta_{(B)}} &\leq \frac{1}{1 + \sqrt{2}} \end{aligned}$$

To violate CHSH inequality:

$$\frac{\eta_{(A)}\eta_{(B)}}{\eta_{(A)} + \eta_{(B)}} > \frac{1}{1 + \sqrt{2}}$$

Suppose Alice and Bob uses same kind of detector i.e. $\eta_{(A)}$= $\eta_{(B)}$=η then the required efficiency must be greater than.

$$\frac{\eta^2}{2\eta} > \frac{1}{1+\sqrt{2}}$$
$$\eta > \frac{2}{1+\sqrt{2}}$$
$$\eta > 82.84\%$$

So, if the efficiency of the detector is greater than 82.84% then the CHSH inequality is violated and the possibility of hidden variables controlling the detection of photons in detectors is ruled out.

6 Conclusion

The basic reason for studying loopholes in Bell's experiments is that the quantum non-locality has such a fundamental significance that it is worth discussing as rigorously as possible. Even implausible alternative explanations like the local realism are worth analysing.

Although Prof. Aspect's experiment [6] successfully closes the locality loophole but in the detection loophole the efficiency required by the photon detectors are difficult to reach. Experiments using ion traps have tested CHSH inequality with detection efficiency close to 100% but these experiments do not address locality loophole. Individually both the loopholes have been successfully closed a long time ago but simultaneously it was achieved only in 2015 at TU Delft[8].

In this they took two artificial diamonds whose lattice was not perfect. It contained a nitrogen atom and next to it there was a missing carbon atom which acted as an electron trap. The trapped electron has a spin that can be denoted as a superposition of spin up and down. The same phenomena happens 1.28 Km away. The distance ensures that the detectors can not exchange information within the time necessary to detect their spin. Now, at both the places the laser excites the electrons that emits photons which is entangled with spin of electron. The entanglement between two distant spins is achieved by 'entanglement swapping'[9] using another location, midway between the other two. The two photons are sent to the mid location where they are overlapped on the beam splitter and detected. The electrons were entangled and measured 245 times. Due to the entanglement, the measurements were highly correlated and Einstein's hidden variable are proved wrong.

Although the arguments against local hidden variable theories exploiting the detector efficiency and locality loophole are strong, experimentalists continue to work towards more definitive tests.

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